Affine GARCH option valuation models

Xize Ye

Western University

iCAIR Online Seminar 2020

iCAIR
international Center for Asset Management,
Insurance, and Risk Management

Overview

- Introduction of option valuation models
- Heston-Nandi GARCH with a variance dependent pricing kernel
  - Motivation
  - Properties
  - Results
- Summary

From Black-Scholes to Stochastic Volatility (SV) models

Black–Scholes model assumes the following dynamics for (physical) stock price

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dW(t). \tag{1}$$

However, the assumption of constant volatility is not realistic. We could otherwise assume a stochastic volatility by replacing σ by σ_t, and give further assumptions on the structure/dynamics of the stochastic volatility process. For example, the Heston (1993) model assumes the following dynamics

$$dS(t) = \mu\, S(t)\, dt + \sqrt{v(t)}\, S(t)\, dW_1(t). \tag{2}$$

$$d\sqrt{v(t)} = -\beta\sqrt{v(t)}\ dt + \delta\ dW_2(t). \tag{3}$$

Affine GARCH models

On the contrary, Affine GARCH models use a GARCH process to model the conditional variance.

GARCH: Generalized Auto-Regressive Conditional Heteroscedasticity.

Features of GARCH models:

- Constant unconditional volatility but time-varying conditional volatility.
- Volatility at time t depends on both past volatilities and past returns.
- Better fitting than continuous-time models.

Affine: Has closed form or quasi-closed form expressions for European call option prices. In contrast, non-Affine models need to compute option price via Monte Carlo simulation. Affine models are really convenient in:

- Computing option prices at a large scale.
- Model calibration using option prices.

An example

Typical setup for Affine GARCH models: Affine GARCH models often begin with assuming the (physical) stock price follows

$$\log(S(t)) = \log(S(t-1)) + r + \lambda h(t) + \sqrt{h(t)}z(t) \tag{4}$$

where:

$r(t)$: daily risk-free rate

$h(t)$: daily conditional variance, follows a particular GARCH process, for example,

Heston-Nandi GARCH (2000) assumes

$$h(t) = \omega + \beta h(t-1) + \alpha \left(z(t-1) - \gamma \sqrt{h(t-1)} \right)^2 \tag{5}$$

λ: risk premium (usually, higher volatility leads to higher price)

$z(t)$: i.i.d. standard normal noise

Risk-neutral process of Heston-Nandi GARCH (2000)

We have seen that the widely used HN-GARCH (2000) model assumes the following process

$$\log(S(t)) = \log(S(t-1)) + r + \lambda h(t) + \sqrt{h(t)}z(t) \tag{6}$$

$$h(t) = \omega + \beta h(t-1) + \alpha\left(z(t-1) - \gamma\sqrt{h(t-1)}\right)^2 \tag{7}$$

With further assumptions, we can write the risk-neutral process

$$\log(S(t)) = \log(S(t-1)) + r - \frac{1}{2}h(t) + \sqrt{h(t)}z^*(t) \tag{8}$$

$$h(t) = \omega + \beta h(t-1) + \alpha\left(z^*(t-1) - \gamma^*\sqrt{h(t-1)}\right)^2 \tag{9}$$

where

$$z^*(t) = z(t) + \left(\lambda + \frac{1}{2}\right)\sqrt{h(t)}, \quad \gamma^* = \gamma + \lambda + \frac{1}{2} \tag{10}$$

Closed form option pricing formula

With this setting, at time t, an European call option with strike price K and matures at T has price

$$\begin{aligned} C =& e^{-r(T-t)} E_t^*[\mathrm{Max}(S(T) - K, 0)] = \frac{1}{2} S(t) \\ &+ \frac{e^{-r(T-t)}}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f^*(i\phi + 1)}{i\phi}\right] d\phi \\ &- K e^{-r(T-t)} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}\right] d\phi\right) \end{aligned} \quad (11)$$

where the generating function $f(\phi)$ takes log-linear form

$$\begin{aligned} f(\phi) =& S(t)^\phi \exp\Bigg[A(t;T,\phi) + B(t;T,\phi)h(t+1) \\ &+ C(t;T,\phi)\left(z(t) - \gamma\sqrt{h(t)}\right)^2\Bigg] \end{aligned} \quad (12)$$

whose coefficients must be computed recursively.

Problems with this setup

With this particular risk-neutral measure:

- HN-GARCH model fits both return and option data well, but the parameters do not match.

Also, some stylized facts observed from financial data:

- Risk-neutral density has bigger tail than physical density.
- Risk-neutral volatility is often greater than physical volatility.
- Options tend to over-react to short-term volatility changes.

As a consequence, we want to find a non-trivial pricing kernel that links the physical measure with risk-neutral measure well, and at the same time, reflect these stylized facts in the model.

Bigger tail of risk-neutral density

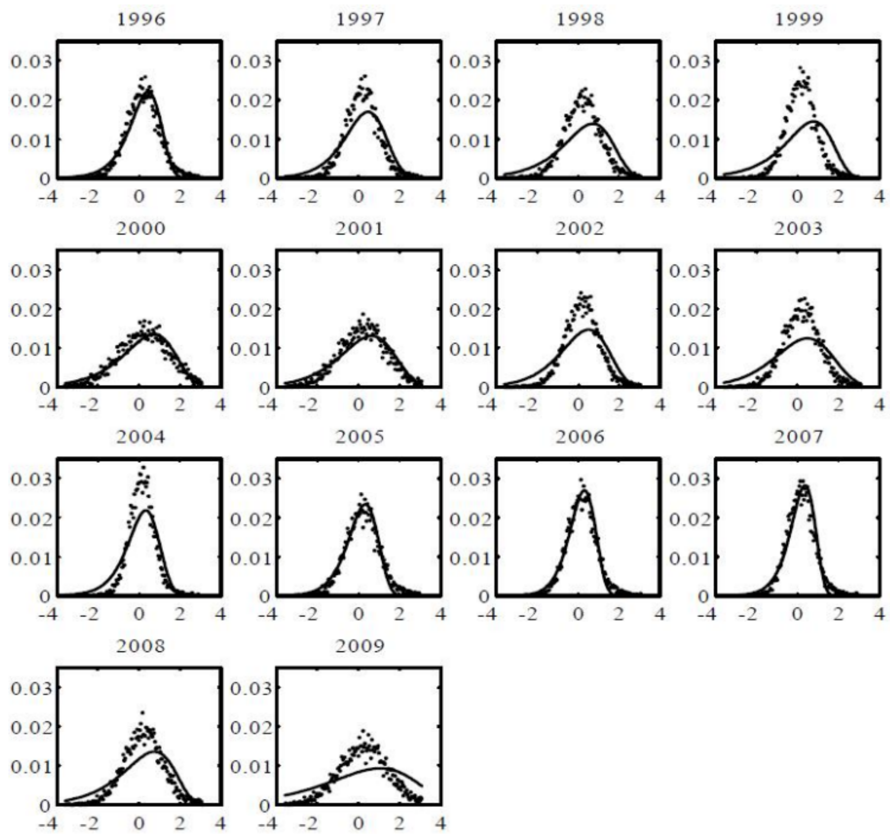

Figure 1. A comparison of risk-neutral densities versus physical GARCH histogram

U shape

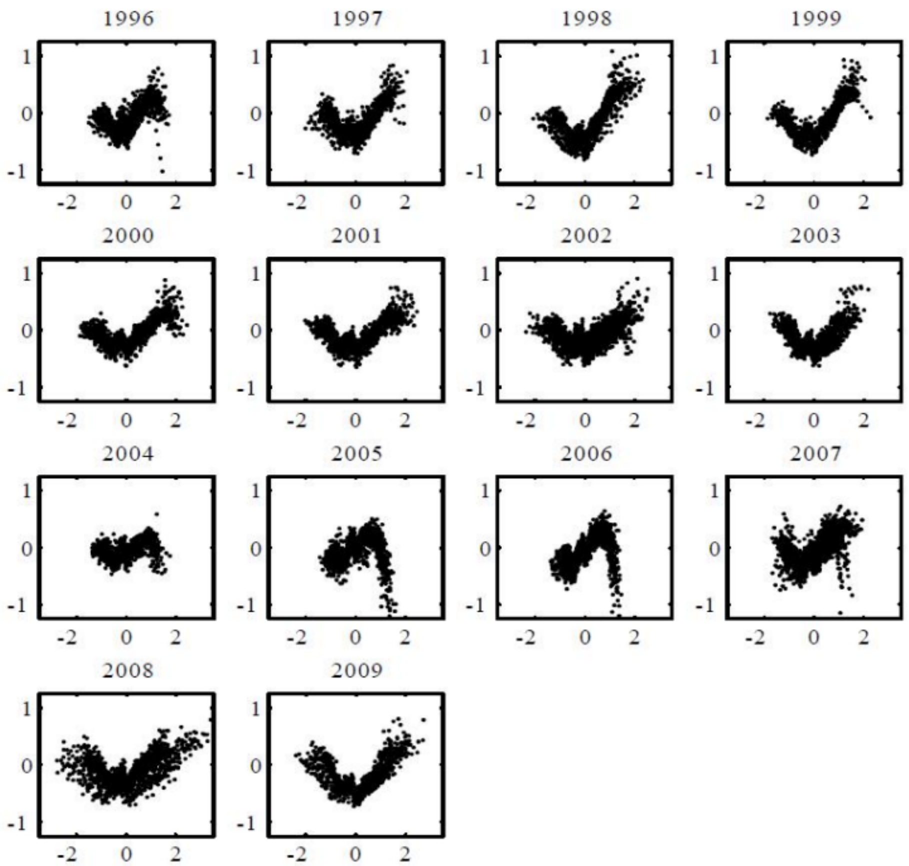

Figure 3. Ratio of densities on a log scale

Higher risk-neutral volatility

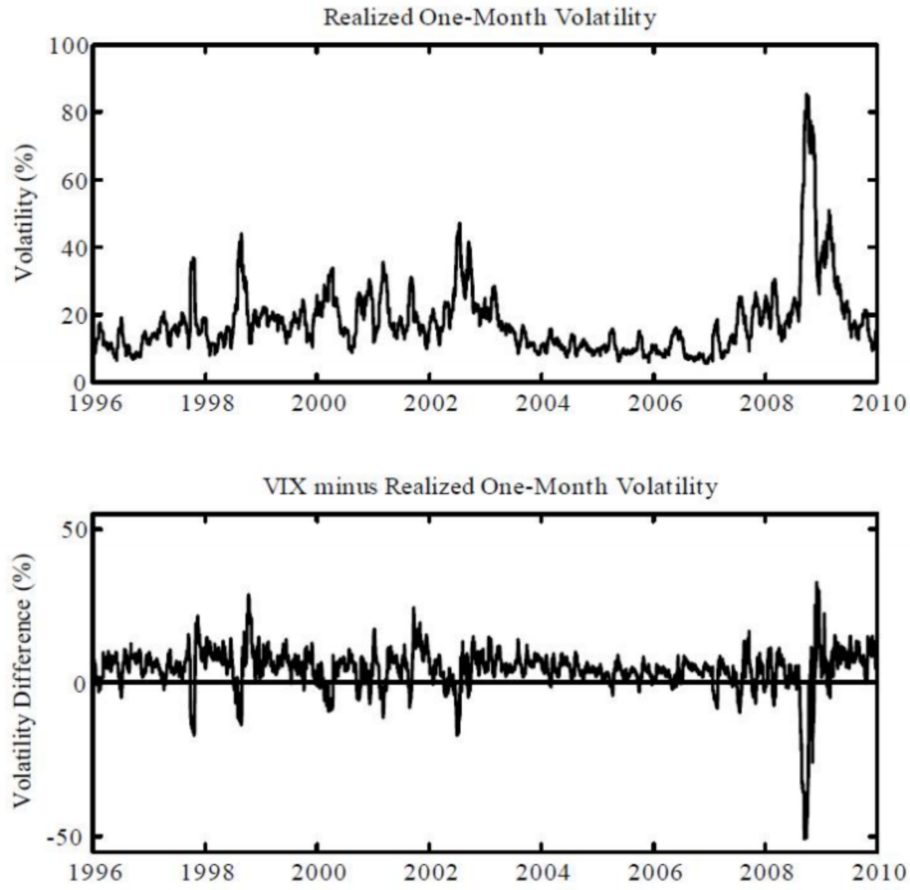

Figure 2. Realized volatility and VIX minus realized volatility

Variance-dependent pricing kernel

To solve these issue, Christoffersen and Heston (2013) proposed a new variance-dependent pricing kernel

$$M(t) = M(0)\left(\frac{S(t)}{S(0)}\right)^{\phi} \exp\left(\delta t + \eta \sum_{s=1}^{t} h(s) + \xi(h(t+1) - h(1))\right) \tag{13}$$

With physical process unchanged, the risk-neutral stock process is

$$\log(S(t)) = \log(S(t-1)) + r - \frac{1}{2}h^*(t) + \sqrt{h^*(t)}z^*(t) \tag{14}$$

$$h^*(t) = \omega^* + \beta h^*(t-1) + \alpha^*\left(z^*(t-1) - \gamma^*\sqrt{h^*(t-1)}\right)^2 \tag{15}$$

where the risk-neutral parameters are

$$\begin{aligned} h^*(t) &= h(t)/(1-2\alpha\xi) \\ \omega^* &= \omega/(1-2\alpha\xi) \\ \alpha^* &= \alpha/(1-2\alpha\xi)^2 \\ \gamma^* &= \gamma - \phi \end{aligned}$$

Implications of this pricing kernel

- When $\xi = 0$, This model corresponds with HN-GARCH (2000).

- When $\lambda > 0.5$, $\xi > 0$, $\gamma > 0$, $h^*(t) > h(t)$, and expected future variance for risk-neutral process exceeds expected future variance for physical process.

- The logarithm of pricing kernel is a quadratic function of log stock return $R(t)$

$$\ln\left(\frac{M(t)}{M(t-1)}\right) = \frac{\xi\alpha}{h(t)}(R(t)-r)^2 - \mu(R(t)-r) + \left(\eta + \xi(\beta-1) + \xi\alpha\left(\mu - \frac{1}{2} + \gamma\right)^2\right)h(t) + \delta + \xi\omega + \phi r \tag{16}$$

Furthermore, when $\xi > 0$, the pricing kernel is U-shaped.

Volatility ratio

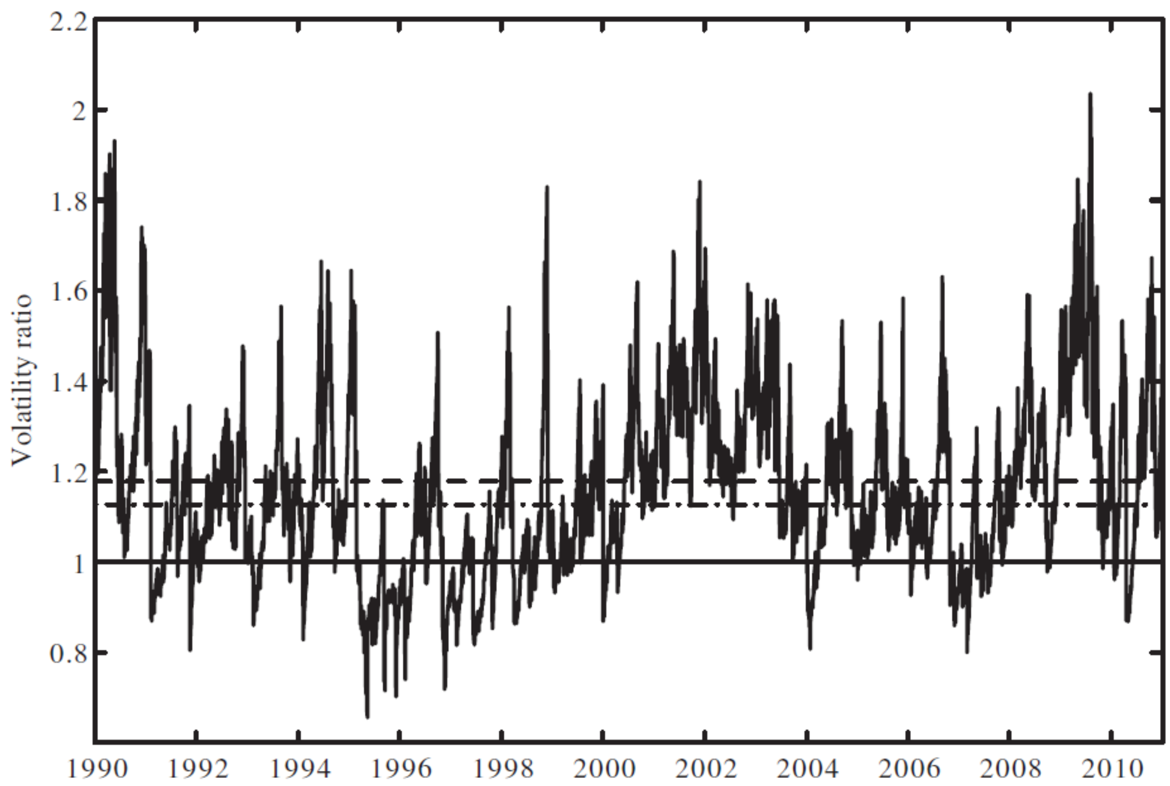

Figure 4. Ratio of risk-neutral and physical volatility

U-shape pricing kernel

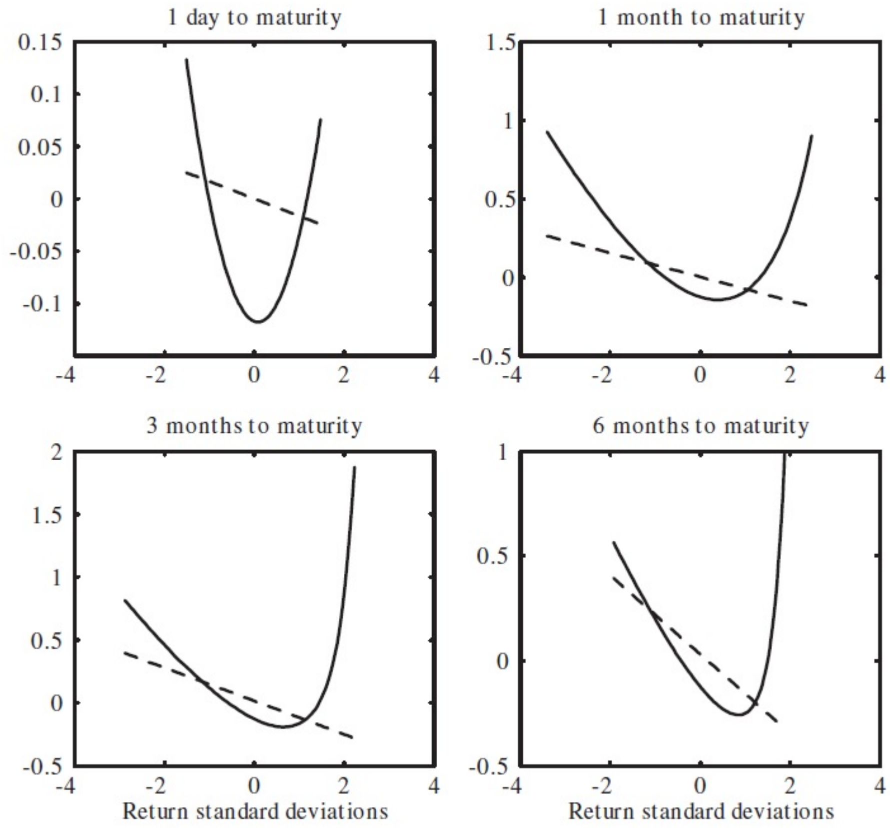

Figure 5. Ratio of risk-neutral densities and physical densities on a log-log scale

Another extension: Multivariate HN-GARCH

One extension for the single-asset HN-GARCH is the Multivariate HN-GARCH with

$$R_t = r\mathbf{1} + \Lambda\Sigma_t\mathbf{1} + A\sqrt{\Sigma_t}z_t \tag{17}$$

$$h_{j,t} = \omega_j + \beta_j h_{j,t-1} + \alpha_j \left(z_{j,t-1} - \gamma_j \sqrt{h_{j,t-1}}\right)^2, \quad j = 1, \ldots, n \tag{18}$$

where:

$r(t)$: daily risk-free rate

R_t: $n \times 1$ vector of log returns of $n-$assets

Λ: $n \times n$ matrix of risk premium such that λ_{ij} is the risk premium effect of the j^{th} asset onto the i^{th} asset

Σ_t: $n \times n$ diagonal matrix that governs the covariance of the multivariate noise z_t

A: $n \times n$ invertible matrix that enforces the correlation between assets.

Summary

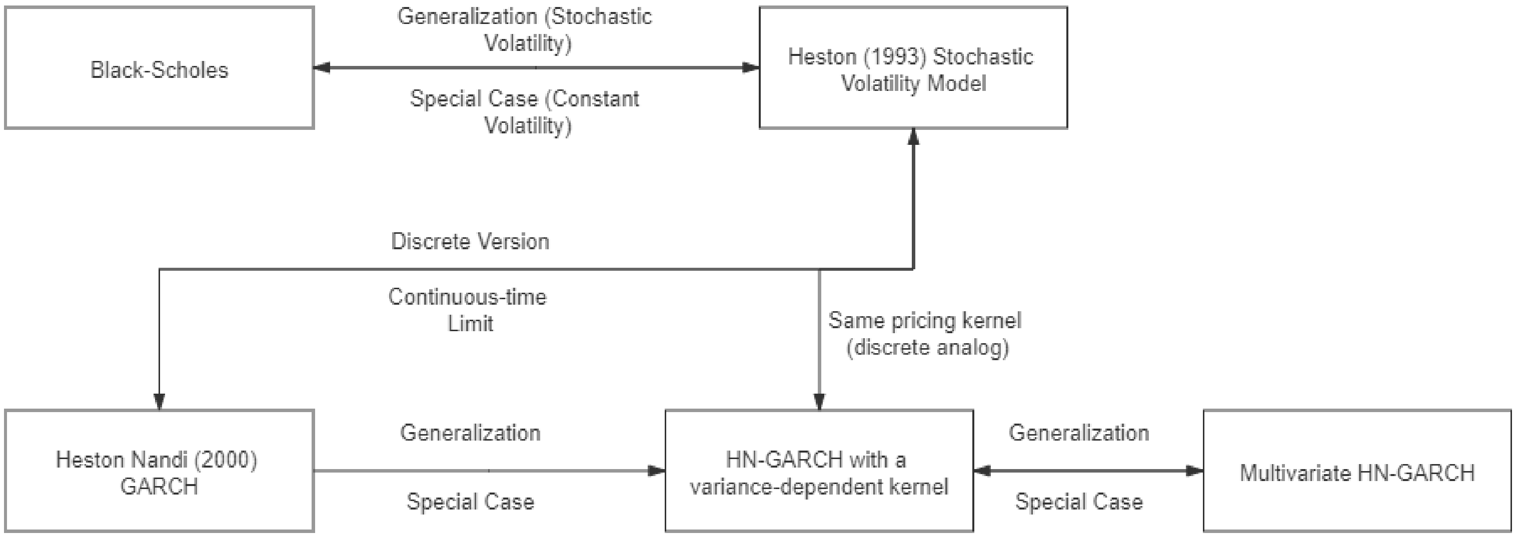

Figure 6. Relation between different option pricing models

References

Steven Heston, Saikat Nandi, A Closed-Form GARCH Option Valuation Model, *The Review of Financial Studies*, Volume 13, Issue 3, July 2000, Pages 585–625, https://doi.org/10.1093/rfs/13.3.585.

Peter Christoffersen, Steven Heston, Kris Jacobs, Capturing Option Anomalies with a Variance-Dependent Pricing Kernel, *The Review of Financial Studies*, Volume 26, Issue 8, August 2013, Pages 1963–2006, https://doi.org/10.1093/rfs/hht033.

Marcos Escobar-Anel, Javad Rastegari, Lars Stentoft, Affine multivariate GARCH models, *Journal of Banking & Finance*, Elsevier, Volume 118.