Generalized Autoregressive Conditionally Stochastic Heteroskedasticity: Motivation and Applications

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Abstract

Typical GARCH models are proven successful in capturing time-varying conditional variance of asset return. Nonetheless, the construction that only 1 innovation drives both the return and variance process make it difficult to reconcile historical return and forward-looking information, such as the VIX (volatility index). Instead, we propose a methodology to add another innovation to GARCH models to allow stochastic volatility, hence resulting in a 2-shock model named Generalized Autoregressive Conditionally Stochastic Heteroskedasticity (GARCSH). In this talk, we discuss the motivation, implementation and financial applications of GARCSH.

1. Intro: option pricing, GARCH and the VIX
2. Motivation of GARCSH
3. GARCSH implementation
4. Numerical Results
5. Conclusion

Introduction: Option pricing

- Key: model time-varying of asset return

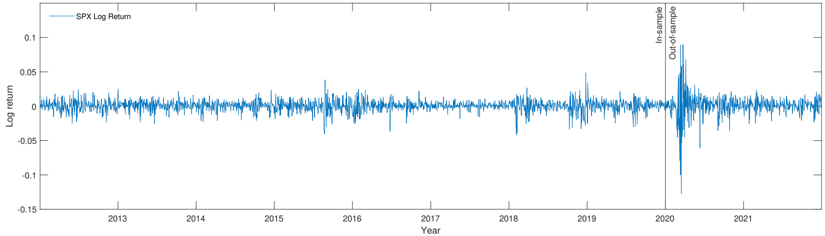


Figure 1: Daily SPX log return

- Two popular strands:
    - Continuous-time stochastic volatility (SV) models. Example includes the Heston (1993) SV model.
    - Discrete-time Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model.

GARCH

GARCH-type processes assume:

$$r_t = \mu_t + \sqrt{h_t} z_t \quad (1)$$

$$h_{t+1} = \omega + \beta h_t + f(z_t) \quad (2)$$

where:

- Subscripts $t = 1, ..., N$ are calendar days
- S_t is asset price (often closing price)
- $r_t := \log(S_{t+1}) - \log(S_t)$ is the daily log return
- μ_t is daily conditional mean return, that includes risk-free return and risk premium
- h_t: conditional variance
- z_t: iid $N(0, 1)$
- f determines the specification of variance dynamics
- Both r_t and h_{t+1} is $\mathscr{F}_t$-measureable

Examples of GARCH

Example of GARCH (or f specification) includes:

$$r_t = \mu_t + \sqrt{h_t} z_t$$
$$h_{t+1} = \omega + \beta h_t + f(z_t)$$

- GARCH(1,1) of Bollerslev (1986): $f(z_t) = \alpha(h_t z_t)^2$
- Affine GARCH of Heston and Nandi (2000), henceforth HN-GARCH: $f(z_t) = \alpha(z_t - \gamma\sqrt{h_t})^2$

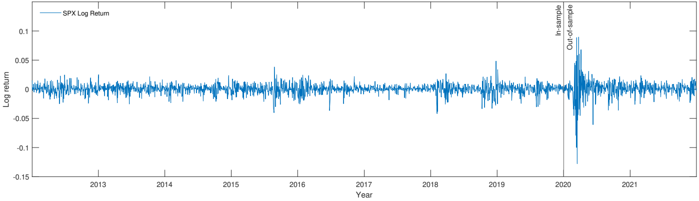
Benefits of GARCH:

- Variance is observable (easily filtered given a GARCH process); as a comparison, variance is latent in SV.
- Easier to implement on daily asset return via maximum likelihood.

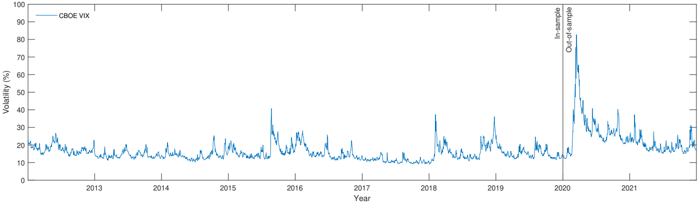
Introduction: the VIX

- GARCH process naturally filters historical conditional variances as $h_1, \ldots, h_T$.
- Alternatively, the Chicago Board Options Exchange (CBOE) provides a model-free volatility index, VIX, that measures expected volatility of S&P 500 over the next 30-day period.
- The VIX is computed from option prices, therefore contains forward-looking information,
- The VIX is quoted in terms of annualized volatility.

SPX and VIX plot



(a) SPX log return



(b) VIX values

GARCH implied VIX

Under GARCH, h_{t+1} denotes the daily conditional variance over $[t, t+1]$. Let $\bar{H}_t$ denote the expected variance over the next τ days. We have

$$\bar{H}_t = \frac{1}{\tau} \sum_{k=1}^{\tau} \mathbb{E}^{\mathbb{Q}}(h_{t+k}) = a h_{t+1} + b,$$

for some constants a and b, i.e. the spot variance and expected future variance is linear. On the other hand, define

$$H_t = \frac{1}{252} \left(\frac{\text{VIX}_t}{100} \right)^2$$

that transforms VIX into daily variance.

Mismatch between GARCH variance and VIX I

Theoretically, H_t from VIX should match GARCH implied variance $\bar{H}_t$, As the graph below shows, HN-GARCH implied VIX cannot match the CBOE VIX, especially under financial crisis.

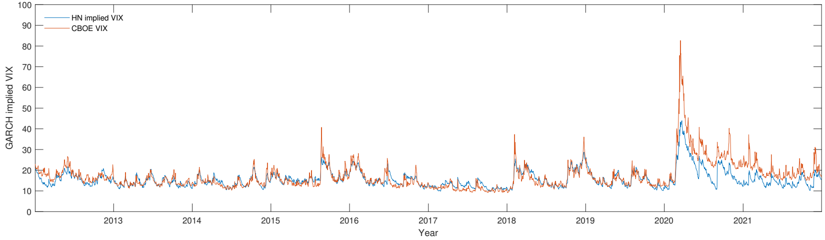


Figure 2: HN implied VIX vs CBOE VIX

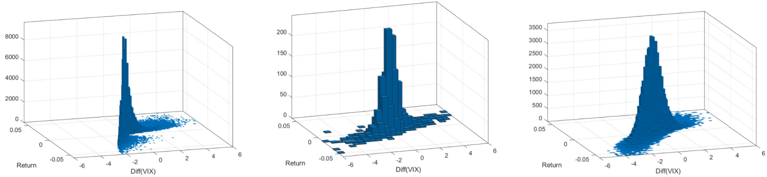
Mismatch between GARCH variance and VIX II

Our finding is not new. Hao and Zhang (2013) empirically tested several GARCH specifications and concluded that when GARCH models are estimated with returns only, the implied variance is consistently lower than VIX. Even when GARCH model is estimated with returns and VIX, the GARCH implied VIX cannot match the VIX.

This leaves room of improvements in the variance dynamics, and pricing kernel.

Motivation of GARCSH: I

If one considers the two time series (r_t) and H_t (or VIX), regular GARCH model assumes both series are driven by the same innovation z_1. The perfect casualty generally doesn't hold given financial data.



(a) From the HN-GARCH model (b) From the empirical data (c) From the HN-GARCSH model

Figure 3: Comparison of joint histogram of return and VIX increments

Motivation of GARCSH: II

The affine HN-GARCH converges weakly (in continuous time) to the Heston model

$$dx_t = (r + \lambda v_t)dt + \sqrt{v_t} dW_t \tag{3}$$

$$dv_t = \kappa(\theta - v_t)dt - \sqrt{v_t}\sigma dW_t, \tag{4}$$

where the asset and variance are driven by the same Brownian motion W_t.

This is not compatible with the continuous-time literature since the full Heston model has two correlated Brownian motions.

Motivation of GARCSH: III

The VIX index also contains other noises, such as measurement errors from

- Option prices
  - Inaccurate prices due to limited liquidity
  - Bid-ask spread
- Implementation of VIX, since the VIX formula contains a few approximations. Jiang and Tian (2007) examined these approximations and concluded the bias in the VIX formula could be substantial.

All this evidence suggests that variance, or VIX, demands its own noise.

GARCSH

The conditionally stochastic component is readily configured on any GARCH model by adding a second innovation

$$r_t = \mu_t + \sqrt{h_t} z_t$$
$$h_{t+1} = \omega + \beta h_t + f(z_t) + g(X_t).$$

Note that either g or X_t should be non-negative to ensure positive variance.

HN-GARCSH

We choose to build the GARCSH component on HN, resulting in HN-GARCSH. Considerations:

- The affine model allows closed-form pricing of asset and VIX derivatives.
- The affine dynamic in HN-GARCH is more restrictive than non-affine GARCH models. Therefore it needs the GARCSH component the most.

The HN-GARCSH model:

$$r_t = \mu_t + \sqrt{h_t} z_t$$
$$h_{t+1} = \omega + \beta h_t + \alpha(z_t - \gamma\sqrt{h_t})^2 + \rho X_t,$$

where X_t is Chi-squared, independent of z_t distributed to maintain the affine structure. Parameter ρ determines the magnitude of the second noise. When $\rho = 0$, HN-GARCSH reduces to HN-GARCH.

HN-GARCSH

The HN-GARCSH model has promising characteristics:

- The two shock model can fit two series simultaneously. One natural choice is the VIX. Therefore, one innovation targets the return, while the other innovation targets the VIX.
- The model maintains the affine structure, allowing for efficient pricing of European options on return, VIX, and also VIX futures. Since Kanniainen et al. (2014) and Chorro and Zazaravaka (2022) highlight the importance of using VIX for improving option pricing performances. In our model, we use the forward-looking information from the VIX explicitly by considering VIX in the modelling.

HN-GARCSH

- The continuous-time limit of HN-GARCSH is Heston with two non-perfectly correlated Brownian motions

$$dx_t = (r + \lambda_1 v_t)dt + \sqrt{v_t}dW_t, \text{ and} \quad (5)$$

$$dv_t = \kappa(\theta - v_t)dt - \sqrt{v_t}(\sigma_1 dW_t + \sigma_2 d\tilde{W}_t). \quad (6)$$

Therefore, the theoretical soundness of and validity of our GARCSH framework is backed when one considers the limiting behavior of discrete-time models to continuous-time.

Decomposition of variance

$$h_{t+1} = \underbrace{\omega + \beta h_t}_{\text{component 1}} + \underbrace{\alpha \left(z_{1,t} - \gamma_1 \sqrt{h_t} \right)^2}_{\text{component 2}} + \underbrace{\rho X_t}_{\text{component 3}} \tag{7}$$

- Component 1: conditionally constant
- Component 2: noise shared with return
- Component 3: variance noise

Table 1: Variance decomposition based on MLEs

Model	C1	C2	C3	Total
HN-GARCH	74.44%	25.56%	0%	100%
HN-GARCSH	0%	50.24%	49.76%	100%

Therefore, GARCSH is intuitively understood as replacing the constant part in variance by a stochastic component.

More flexible filtered variance

Below compares the filtered variance of HN-GARCH and HN-GARCSH. Recall that when market is stable, HN-GARCH will underprice variance significantly. On the otherhand, HN-GARCSH utilizes VIX information to make up for this underpricing. The more dynamical variance structure helps capture sudden jumps and spikes in the VIX.

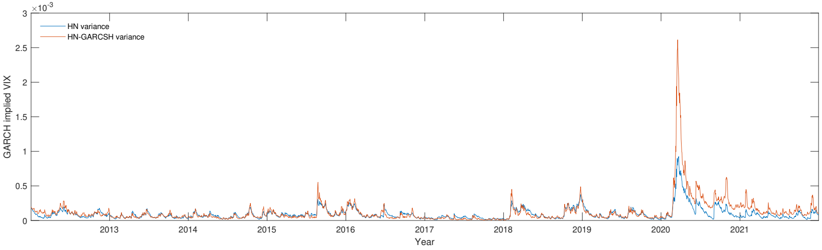


Figure 4: Comparison of filtered variance

Option pricing result

Table 2: Option pricing result

Model	HN	HN-GARCSH
Panel A: SPX options pricing		
In-sample		
Mean of IV error	5.033	1.335
RMSE of IV error	7.634	5.208
Out-of-sample		
Mean of IV error	12.377	3.552
RMSE of IV error	15.732	9.708
Panel B: VIX futures pricing		
In-sample		
Mean error	2.657	-1.036
RMSE	3.721	1.847
Out-of-sample		
Mean error	11.604	1.959
RMSE	13.263	3.336
Panel C: VIX options pricing		
In-sample		
Mean error	1.115	0.083
RMSE	1.543	1.044
Out-of-sample		
Mean error	2.624	2.032
RMSE	3.166	2.516

Conclusion

- We propose a GARCSH framework that is readily applicable to all GARCH models.
- We choose to implement on the affine HN-GARCH model. The resulting HN-GARCSH model:
  - has closed-form solutions in asset and VIX derivatives
  - has a dynamical variance process
  - produces much better fit to VIX and option prices

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Q & A

Thanks for your attention. Any questions?