Generalized Autoregressive Conditionally Stochastic Heteroskedasticity: Motivation and Applications

Xize Ye @ CSSC 2024

Department of Statistical and Actuarial Sciences
Western University

1. Intro: option pricing, GARCH and the VIX
2. GARCH vs GARCSH
3. Numerical Results
4. Conclusion

Introduction: Option pricing

- Key: model time-varying volatility of asset return

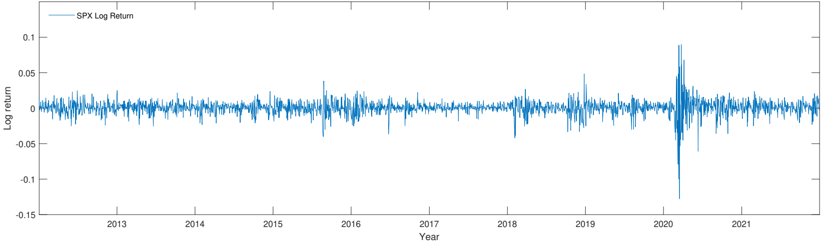


Figure 1: Daily SPX log return

- Two popular strands:
  - Continuous-time stochastic volatility (SV) models. Example includes the Heston (1993) SV model.
  - Discrete-time Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model.

HN-GARCH

HN-GARCH of Heston and Nandi (2000) assume (t in days):

$$r_t = \mu_t + \sqrt{h_t} z_t \tag{1}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(z_t - \gamma\sqrt{h_t})^2. \tag{2}$$

Under GARCH, h_{t+1} denotes the daily conditional variance over $[t, t+1]$. Let $\bar{H}_t$ denote the expected variance over the next τ days. We have

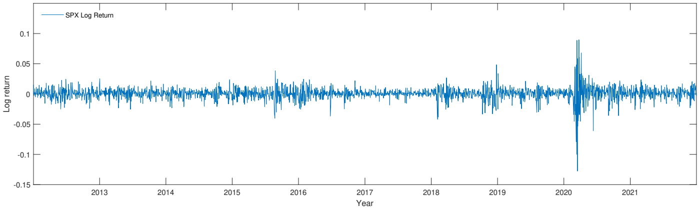
$$\bar{H}_t = \frac{1}{\tau}\sum_{k=1}^{\tau} \mathbb{E}^{\mathbb{Q}}(h_{t+k}) = a h_{t+1} + b,$$

for some constants a and b, i.e., the spot variance and expected future variance is linear.

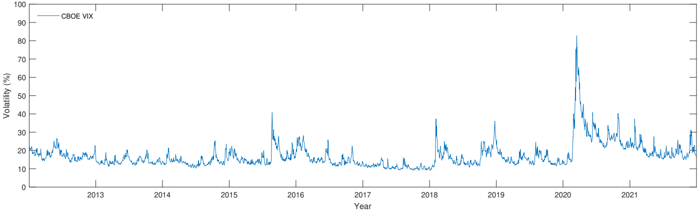
Introduction: the VIX

On the other hand, the Chicago Board Options Exchange (CBOE) provides a model-free volatility index, VIX, that measures the expected variance of S&P 500 over the next 30-day period (quoted in annualized volatility).

SPX and VIX plot



(a) SPX log return



(b) VIX values

Mismatch between GARCH variance and VIX I

Theoretically, GARCH implied future variance and VIX should match. Empirically, as the graph below shows, HN-GARCH implied VIX cannot match the CBOE VIX, especially under financial crisis.

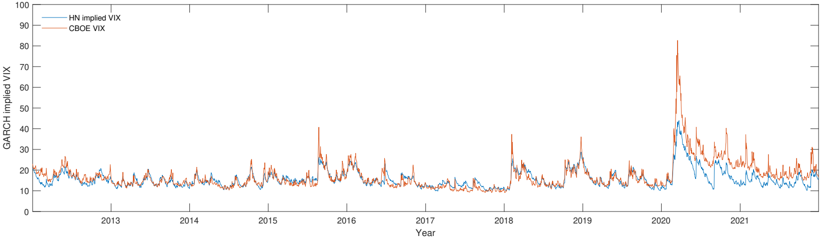


Figure 2: HN implied VIX vs CBOE VIX

This leaves room for improvements in the variance dynamics, and pricing kernel.

HN-GARCSH

The HN-GARCSH model:

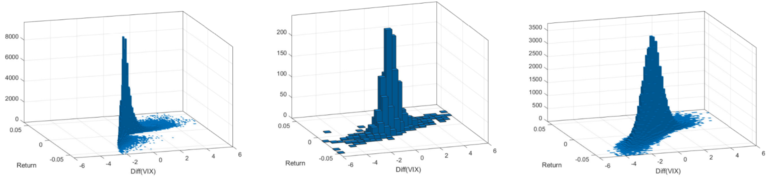
$$r_t = \mu_t + \sqrt{h_t} z_t$$
$$h_{t+1} = \omega + \beta h_t + \alpha(z_t - \gamma\sqrt{h_t})^2 + \rho X_t,$$

where X_t is Chi-squared, independent of z_t distributed to maintain the affine structure. Parameter ρ determines the magnitude of the second noise. When $\rho = 0$, HN-GARCSH reduces to HN-GARCH. In our implementation, we choose to build the GARCSH component upon HN because

- the affine model allows closed-form pricing of asset and VIX derivatives.
- the affine dynamic in HN-GARCH is more restrictive than non-affine GARCH models. Therefore it will benefit more from having the GARCSH component.

GARCH vs GARCSH: I

If one considers the two time series (r_t) and H_t (or VIX), regular GARCH model assumes both series are driven by the same innovation z_1. The perfect casualty generally doesn't hold given financial data.



(a) From the HN-GARCH model (b) From the empirical data (c) From the HN-GARCSH model

Figure 3: Comparison of joint histogram of return and VIX increments

GARCH vs GARCSH: II

The affine HN-GARCH converges weakly (in continuous time) to the Heston model

$$dx_t = (r + \lambda v_t)dt + \sqrt{v_t}dW_t \quad (3)$$

$$dv_t = \kappa(\theta - v_t)dt - \sqrt{v_t}\sigma dW_t, \quad (4)$$

where the asset and variance are driven by the same Brownian motion W_t.

On the other hand, the continuous-time limit of HN-GARCSH is Heston with two non-perfectly correlated Brownian motions

$$dx_t = (r + \lambda_1 v_t)dt + \sqrt{v_t}dW_t, \text{ and} \quad (5)$$

$$dv_t = \kappa(\theta - v_t)dt - \sqrt{v_t}(\sigma_1 dW_t + \sigma_2 d\tilde{W}_t). \quad (6)$$

Therefore, our proposal is backed when considering the limiting behaviour.

Decomposition of variance

$$h_{t+1} = \underbrace{\omega + \beta h_t}_{\text{component 1}} + \underbrace{\alpha \left(z_{1,t} - \gamma_1 \sqrt{h_t} \right)^2}_{\text{component 2}} + \underbrace{\rho X_t}_{\text{component 3}} \tag{7}$$

- Component 1: conditionally constant
- Component 2: noise shared with return
- Component 3: variance noise

Table 1: Variance decomposition based on MLEs

Model	C1	C2	C3	Total
HN-GARCH	74.44%	25.56%	0%	100%
HN-GARCSH	0%	50.24%	49.76%	100%

Therefore, GARCSH is intuitively understood as replacing the constant part in variance by a stochastic component.

More flexible filtered variance

HN-GARCSH component can help capture jump and sudden spikes in the VIX and make up for the underpricing ordinary GARCH models would suffer.

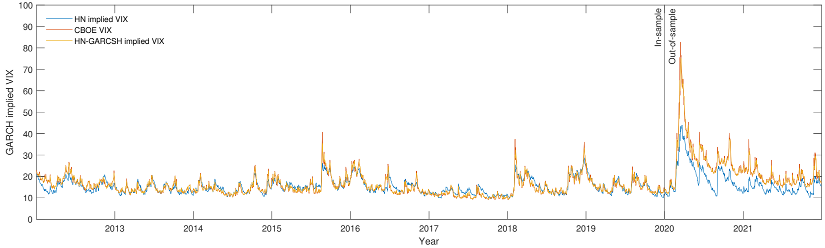


Figure 4: Comparison of implied VIX

Option pricing result

Table 2: Option pricing result

Model	HN	HN-GARCSH	Reduction (%)
Panel A: SPX options pricing			
In-sample			
Mean of IV error	5.033	1.335	73.5
RMSE of IV error	7.634	5.208	31.8
Out-of-sample			
Mean of IV error	12.377	3.552	71.3
RMSE of IV error	15.732	9.708	38.3
Panel B: VIX futures pricing			
In-sample			
Mean error	2.657	-1.036	61.0
RMSE	3.721	1.847	50.4
Out-of-sample			
Mean error	11.604	1.959	83.1
RMSE	13.263	3.336	74.8
Panel C: VIX options pricing			
In-sample			
Mean error	1.115	0.083	92.6
RMSE	1.543	1.044	32.3
Out-of-sample			
Mean error	2.624	2.032	22.6
RMSE	3.166	2.516	20.5

Conclusion

- We propose the GARCSH framework, a generalization that is readily applicable to all GARCH models.
- We provide an example that combines GARCSH component on the affine HN-GARCH model.
- The resulting HN-GARCSH model
  - has closed-form solutions in asset and VIX derivatives
  - has a more dynamic variance process
  - produces better fit simultaneously to return, VIX, and option prices

References I

Heston, S. L. (1993), 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', *The Review of Financial Studies* **6**(2), 327–343.

Heston, S. L. and Nandi, S. (2000), 'A closed-form GARCH option valuation model', *The Review of Financial Studies* **13**(3), 585–625.

Q & A

Thank you very much for listening!!

Any questions?